Last Time: Predicates, Quantifiers, Sets.

Today: More...

Recall: A predicate is a statement w/ variables.

The variables represent elements of a domain set. ("subjects")

Ex: $L(x,y)$: "x loves y"

Domain: $\{z : z \text{ is a person}\}$

A quantifier tells us how many subjects satisfy a predicate statement (roughly...)

Universal Quantifier: $\forall x\, P(x)$ = "for all x, P(x)"

Existential Quantifier: $\exists x\, P(x)$ = "At least one x satisfies P(x)"

i.e. "There exists an x satisfying P(x)"

Uniqueness Quantifier: $\exists ! (x \in \mathbb{R})\, P(x)$ = "There is a unique x satisfying P(x)"

Unique means "exactly one"

So "$\exists !$" reads "there is one and only one"

↕ The Highlander Quantifier...

There can be only one...

Ex: Translate the statement:

"There is a unique natural number, n, so that for all natural numbers, m, $n+m=m$"

Sol: We rewrite

"There exists a unique $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ we have $n+m=m$"

- $\mathbb{N}$: set of natural numbers
- $n \in \mathbb{N}$: quantifier
- "unique": "scoping" quantifier
- "such that": "scoping"
- "for all $m \in \mathbb{N}$": quantifier
- "we have": "scoping"
- $n+m=m$: Predicate

Now this becomes: $\exists! n \in \mathbb{N}\left[\forall m \in \mathbb{N}[n+m=m]\right]$. ∎

Q: How does negation interact w/ Quantification?

A: Note that $\forall x \in D[P(x)]$ means "every x in D satisfies $P(x)$". If $D = \{d_1, d_2, \cdots, d_n\}$ then $\forall x \in D[P(x)]$ is equivalent to $P(d_1) \wedge P(d_2) \wedge \cdots \wedge P(d_n)$.

So $\neg \forall x \in D[P(x)]$ would mean $(\neg P(d_1)) \vee (\neg P(d_2)) \vee \cdots \vee (\neg P(d_n))$

OTOH $\exists x \in D[Q(x)]$ would mean (for finite D):

$$Q(d_1) \vee Q(d_2) \vee \cdots \vee Q(d_n)$$

Point: $\forall$ ⇝ "big and" $\exists$ ⇝ "big or"

So: $\neg \forall_x P(x) \equiv \exists x[\neg P(x)]$

and $\neg \exists_x P(x) \equiv \forall x[\neg P(x)]$

De Morgan's Quantifier Laws

Ex: Negate $\exists_{x \in \mathbb{N}} \forall_{y \in \mathbb{N}} [x + y = y]$.

Sol: $\neg \exists_{x \in \mathbb{N}} \forall_{y \in \mathbb{N}} [x + y = y]$

$\equiv \forall_{x \in \mathbb{N}} [\neg \forall_{y \in \mathbb{N}} [x + y = y]]$ (De Morgan)

$\equiv \forall_{x \in \mathbb{N}} \exists_{y \in \mathbb{N}} [\neg (x + y = y)]$ ▨

Set Theory

A set is a collection of elements (i.e. objects in the set).

We write $x \in S$ to mean "x is an element of S"

Two sets are equal when they have the same elements.

(So: order and repeated elements don't change the set...)

Notations: between the braces ...

Roster notation

List out all elements

Ex: $\{1, 2, 3, 5, 12, -15\}$

Set-Builder

Describe the elements

Ex: $\{x \in \mathbb{N} : x \text{ is even}\}$.

Logically equivalent predicates give the same set...

Set Operations

Goal: Build new sets from old...

Running Example: Let $A = \{1, 2, -3, 5, -7\}$,
$B = \{-1, 4, 5\}$, $C = \{1, 5, -7\}$.

Def^n: The **cardinality** of set S is the number of elements it contains. We denote the cardinality of S by $|S|$ or $\#S$.

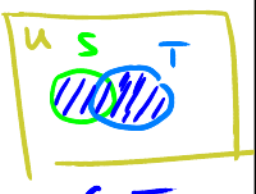
Ex: $|A| = |\{1, 2, -3, 5, -7\}| = 5 = \#A$,

$|B| = |\{-1, 4, 5\}| = 3$, $|C| = |\{1, 5, -7\}| = 3$

"equicardinal" i.e. have same cardinality.

Def^n: The **union** of sets S and T is

$$S \cup T := \{x : x \in S \text{ or } x \in T\}.$$



Ex: $A \cup B = \{1, 2, -3, 5, -7\} \cup \{-1, 4, 5\}$

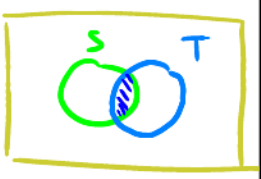
$= \{1, 2, -3, 5, -7, -1, 4, 5\}$

$= \{1, 2, -3, 5, -7, -1, 4\}$

$B \cup C = \{-1, 4, 5\} \cup \{1, 5, -7\} = \{-1, 4, 5, 1, -7\}$

Def: The intersection of sets S and T is

$$S \cap T = \{x : x \in S \text{ and } x \in T\}$$



Ex: $A \cap B = \{1, 2, -3, 5, -7\} \cap \{-1, 4, 5\} = \{5\}$

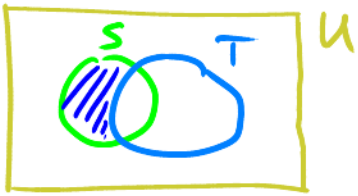
"singleton" b/c it has cardinality 1.

Ex: $\{1, 3, 7\} \cap \{2, 4, 8\} = \{\} = \emptyset$

"empty set" or "null set"

Defn: The set difference, S remove T is

$$S \setminus T = \{x : x \in S \text{ but } x \notin T\}$$



Ex: $A \setminus B = \{1, 2, -3, 5, -7\} \setminus \{-1, 4, 5\}$

$$= \{1, 2, -3, -7\}$$

$$C \setminus B = \{1, 5, -7\} \setminus \{-1, 4, 5\} = \{1, -7\}$$

Defn: The cartesian product of set S with T is

$$S \times T = \{(x, y) : x \in S \text{ and } y \in T\}$$

Ex: Order Matters for cartesian product!!! $B \times C \neq C \times B$.

$$B \times C = \{-1, 4, 5\} \times \{1, 5, -7\}$$

$$= \{(-1, 1), (-1, 5), (-1, -7), (4, 1), (4, 5), (4, -7), (5, 1), (5, 5), (5, -7)\}$$

$$C \times B = \{1, 5, -7\} \times \{-1, 4, 5\}$$

$$= \{(1, -1), (1, 4), (1, 5), (5, -1), (5, 4), (5, 5), (-7, -1), (-7, 4), (-7, 5)\}$$

Defn: The power set of set S is

$$\mathbb{P}(S) = \{ x : x \subseteq S \}$$

Recall: we write $X \subseteq Y$ for the statement $\forall x \in X [x \in Y]$

i.e. $\forall x [x \in X \rightarrow x \in Y]$.

Ex: $\mathbb{P}(C) = \{ X : X \subseteq C \}$ $\quad C = \{1, 5, -7\}$

$$= \{ \{\}, \{1\}, \{5\}, \{-7\}, \{1,5\}, \{1,-7\}, \{5,-7\}, \{1,5,-7\} \}$$

$\{\}$: card 0; $\{1\}, \{5\}, \{-7\}$: card 1; $\{1,5\}, \{1,-7\}, \{5,-7\}$: card 2; $\{1,5,-7\}$: card 3

$$\mathbb{P}(\emptyset) = \mathbb{P}(\{\}) = \{ \{\} \}. \quad \text{Not: } |\mathbb{P}(\emptyset)| = 1$$

$\{\}$: card 0; $\{\} = \{\emptyset\}$; $\{\{\}\}$: card 1

Next Time: Proofs involving sets!